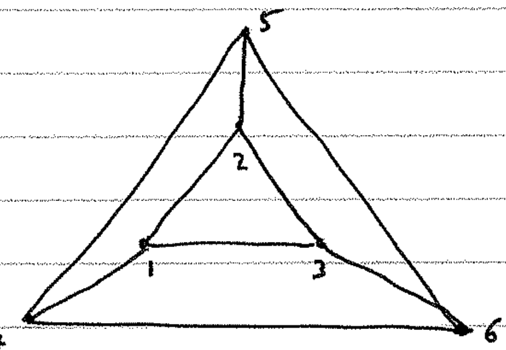
Problem Set #5, Solutions

1 (a) From Eq. 26 we list all pairs of connected vertices:

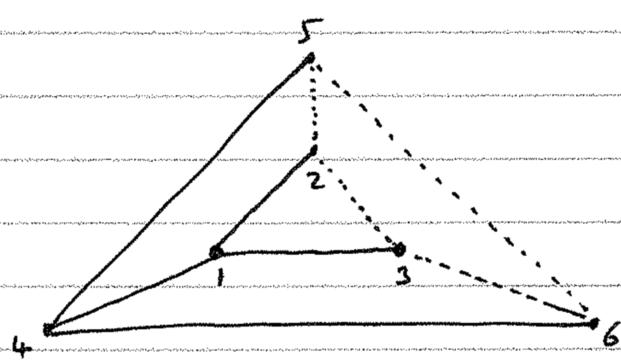
$(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,5)$,
$(3,6)$, $(4,5)$, $(4,6)$, $(5,6)$

network:



$N = 6$
$E = 9$
$C = 9 - 6 + 1$
$= 4$

One choice for maximal tree (solid lines):



Corresponding fundamental cycles:

$$\begin{cases} c_1 = 1231 \\ c_2 = 14521 \\ c_3 = 13641 \\ c_4 = 25632 \end{cases}$$

→

(b) Affinities:

$$A_{c_1} = \ln \frac{R_{21} R_{32} R_{13}}{R_{12} R_{23} R_{31}} = \ln(\sigma \cdot \sigma \cdot \sigma) = -\frac{3fd}{T_b}$$

(Eq. 27)

During cycle c_1, the system (or "particle") moves 3 steps to the right, in Mode 1. Its energy thus increases by $+3df$ (Fig. 6). This energy comes from reservoir B, hence

$$\Delta S = \frac{-3fd}{T_B} = A_{c_1} \quad \checkmark$$

$$A_{c_2} = \ln \frac{R_{41} R_{54} R_{25} R_{12}}{R_{14} R_{45} R_{52} R_{21}} = \ln\left(\frac{1}{\mu} \cdot \nu\sigma \cdot 1 \cdot \frac{1}{\sigma}\right)$$

$$= \frac{\alpha}{T_A} - \frac{\alpha}{T_B} \quad \text{(Eq. 5)}$$

$1 \to 4$: particle releases energy α to res. A

$4 \to 5$: particle absorbs $\alpha + fd$ from res. B

$5 \to 2$: no exchange of energies

$2 \to 1$: particle releases energy fd to res B

$$\Delta S = \frac{\alpha}{T_A} - \frac{\alpha + fd}{T_B} + \frac{fd}{T_B} = \frac{\alpha}{T_A} - \frac{\alpha}{T_B} = A_{c_2}$$

$\checkmark$

$$A_{C_3} = \ln \frac{R_{31} R_{63} R_{46} R_{14}}{R_{13} R_{36} R_{64} R_{41}}$$

$$= \ln\left(\frac{1}{\sigma} \cdot \mu \cdot \frac{\sigma}{\nu^2} \cdot \mu\right) = \frac{2\alpha}{T_B} - \frac{2\alpha}{T_A}$$

$1 \to 3$: particle releases fd to B
$3 \to 6$: particle absorbs α from A
$6 \to 4$: particle releases $2\alpha - fd$ to B
$4 \to 1$: particle absorbs α from A

$$\Delta S = \frac{-2\alpha}{T_A} + \frac{2\alpha}{T_B} = A_{C_3}$$

$$A_{C_4} = \ln \frac{R_{52} R_{65} R_{36} R_{23}}{R_{25} R_{56} R_{63} R_{32}} = \ln\left(1 \cdot \nu\sigma \cdot \frac{1}{\mu} \cdot \frac{1}{\sigma}\right)$$

$$= \frac{\alpha}{T_A} - \frac{\alpha}{T_B}$$

$2 \to 5$: $\emptyset$
$5 \to 6$: particle absorbs $\alpha + fd$ from B
$6 \to 3$: particle releases α to A
$3 \to 2$: particle releases fd to B

$$\Delta S = \frac{\alpha}{T_A} - \frac{\alpha}{T_B} = A_{C_4}$$

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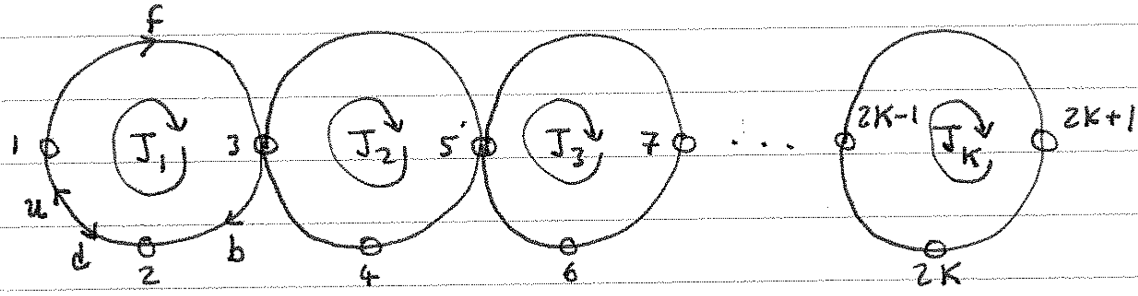

$$J_1 = \pi_2 u - \pi_1 d = \pi_1 f = \pi_3 b$$

similarly: $$J_k = \pi_{2k} u - \pi_{2k-1} d = \pi_{2k-1} f = \pi_{2k+1} b$$

$$\therefore \quad \frac{\pi_{2k+1}}{\pi_{2k-1}} = \frac{f}{b}$$

$$\pi_{2k} = \frac{1}{u}\left(d+f\right)\pi_{2k-1} \rightarrow \frac{\pi_{2k-1}}{\pi_{2k}} = \frac{u}{d+f}$$

$$\therefore \quad \frac{\pi_{2K+1}}{\pi_2} = \underbrace{\frac{\pi_{2K+1}}{\pi_{2K-1}} \; \frac{\pi_{2K-1}}{\pi_{2K-3}} \cdots \frac{\pi_3}{\pi_1}}_{K \text{ factors}} \cdot \frac{\pi_1}{\pi_2}$$

$$= \left(\frac{f}{b}\right)^K \frac{u}{d+f}$$

Problem 3

$$\dot{x} = -\frac{k}{\gamma}(x-ut) + \xi$$

$$\dot{w} = -ku(x-ut)$$

(a) define $y = x - ut$

$$\begin{cases} \dot{y} = \dot{x} - u = -\frac{k}{\gamma} y - u + \xi \\ \dot{w} = -kuy \end{cases}$$

$f(y, w, t)$

$$\frac{\partial f}{\partial t} = \frac{k}{\gamma}\frac{\partial}{\partial y}(yf) + u\frac{\partial f}{\partial y} + D\frac{\partial^2 f}{\partial y^2} + kuy\frac{\partial f}{\partial w}$$

$$g(\lambda, \varphi, t) = \ln \int f(y,w,t)\, e^{\lambda y + \varphi w}$$

$$\frac{\partial g}{\partial t} = e^{-g}\int \frac{\partial f}{\partial t} e^{\lambda y + \varphi w}$$

$$= e^{-g}\left[-\frac{k}{\gamma}\lambda\frac{\partial}{\partial \lambda}(e^g) - u\lambda(e^g) + D\lambda^2 e^g - ku\varphi\frac{\partial}{\partial\lambda}(e^g)\right]$$

$$= -\frac{k}{\gamma}\lambda\frac{\partial g}{\partial \lambda} - u\lambda + D\lambda^2 - ku\varphi\frac{\partial g}{\partial \lambda}$$

$$\cancel{g = \sum_{j,k=0}^{\infty} \lambda^j \varphi^k}$$

$$g = \sum_{m,n=0}^{\infty}\frac{\lambda^m \varphi^n}{m!\,n!}\,\omega_{mn}$$

→

Substituting this expression for g into the equation of motion, & arranging terms by powers $\lambda^m \varphi^n$, we get:

for $m+n=1$:

$$\begin{cases} \dot{\omega}_{10} + \frac{k}{\gamma}\omega_{0} = -u \\ \dot{\omega}_{01} + ku\,\omega_{10} = 0 \end{cases}$$

$m+n=2$:

$$\begin{cases} \dot{\omega}_{20} + \frac{2k}{\gamma}\omega_{20} = 2D \\ \dot{\omega}_{11} + \frac{k}{\gamma}\omega_{11} + ku\,\omega_{20} = 0 \\ \dot{\omega}_{02} + 2ku\,\omega_{11} = 0 \end{cases}$$

For $m+n \geqslant 3$ we get homogeneous coupled equations, i.e. the right side of each equation is zero

[e.g. $\dot{\omega}_{21} + \frac{2k}{\gamma}\omega_{21} + ku\,\omega_{30} = 0$]

Since $\omega_{mn}(0) = 0$ for all m, n such that $m+n \geqslant 3$,

we conclude that

$$\omega_{mn}(t) = 0 \quad \forall t \text{ when } m+n \geqslant 0$$

This implies that $f(y, w, t)$ is a 2-dim'l Gaussian.

Solving the five equations for $m+n \leq 2$, we get:

$$\langle y \rangle = \omega_{10} = -\frac{\gamma u}{k}\left(1 - e^{-kt/\gamma}\right)$$

$$\rightarrow \therefore \langle x \rangle = ut - \frac{\gamma u}{k}\left(1 - e^{-kt/\gamma}\right)$$

$$\rightarrow \langle w \rangle = \omega_{01} = \gamma u^2 \left[t + \frac{\gamma}{k}\left(e^{-kt/\gamma} - 1\right)\right]$$

$$\rightarrow \sigma_x^2 = \sigma_y^2 = \frac{\gamma D}{k} = \frac{1}{\beta k}$$

$$\rightarrow \begin{cases} C_{11} = \langle (x - \langle x \rangle)(w - \langle w \rangle) \rangle \\ \quad = \langle (y - \langle y \rangle)(w - \langle w \rangle) \rangle \\ \quad = \omega_{11} = -\frac{\gamma^2 u D}{k}\left(1 - e^{-kt/\gamma}\right) \end{cases}$$

$$\rightarrow \sigma_w^2 = \omega_{02} = 2\gamma^2 u^2 D \left[t + \frac{\gamma}{k}\left(e^{-kt/\gamma} - 1\right)\right]$$

(b) $$p(w,t) = \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-(w - \langle w \rangle)^2 / 2\sigma_w^2}$$

with $\langle w \rangle$ & σ_w^2 as given above.

Note that $\langle w \rangle = \frac{1}{2\gamma D} \sigma_w^2 = \frac{\beta}{2}\sigma_w^2$

It is easily verified that $\langle e^{-\beta w} \rangle = 1$ when the work distribution is a Gaussian whose mean and variance satisfy $\langle w \rangle = \beta\sigma_w^2/2$.